

Recoupling Coefficients and Quantum Entropies

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In this work, we show that the asymptotic limit of the recoupling coefficients of the symmetric group is characterized by the existence of quantum states of three particles with given eigenvalues for their reduced density matrices. This parallels Wigner’s observation that the semiclassical behavior of the $6j$ -symbols for $SU(2)$ —fundamental to the quantum theory of angular momentum—is governed by the existence of Euclidean tetrahedra. We explain how to deduce solely from symmetry properties of the recoupling coefficients the strong subadditivity of the von Neumann entropy, first proved by Lieb and Ruskai, and discuss possible generalizations of our result.

Spin, the quantum-mechanical analog of angular momentum, is mathematically given by an irreducible representation of the group $SU(2)$, the covering group of the rotations in three-dimensional space. In order to compute the total spin of two particles, one needs to decompose the tensor product of the individual spins into irreducible representations. This decomposition is described by a unitary matrix whose entries are known as the Clebsch–Gordan coefficients, or, equivalently, as the Wigner $3j$ -symbols, which only differ in their normalisation. Clebsch–Gordan coefficients are fundamental to quantum theory, governing, for instance, optical transitions in atoms and molecules. When computing the total spin of three spins j_1 , j_2 and j_3 , one can either start by decomposing j_1 and j_2 to obtain j_{12} and then further decompose j_{12} and j_3 into j_{123} , or, alternatively, decompose j_2 and j_3 into j_{23} and then decompose j_1 and j_{23} into j_{123} . The entries of the unitary matrix relating these two decompositions are known as the *recoupling coefficients*, or as the *Wigner $6j$ -symbols* in their rescaled, more symmetric form [1]; they depend only on the six mentioned spins (the Racah W -coefficients [2] are also closely related). The semiclassical limit where all spins are simultaneously rescaled by $k \rightarrow \infty$ was first considered by Wigner [1] who noted that the absolute value squared of the $6j$ -symbol—corresponding to the probability of particles two and three having total spin j_{23} given the spins $j_1, j_2, j_3, j_{12}, j_{123}$ —oscillates around the inverse volume of the tetrahedron whose edges have length equal to the six spins if such a tetrahedron exists; in particular, it then decays polynomially with k . If no such tetrahedron exists then the $6j$ -symbol decays exponentially. A more precise formula together with a heuristic proof was given by Ponzano and Regge [3] and only proved in 1999 by Roberts [4]. This formula has recently been studied in depth in the context of quantum gravity, more precisely, in connection with spin foams and spin networks (see e.g. [5–10]). Wigner $6j$ -symbols have also appeared in quantum information theory [11, 12].

In this paper we consider the recoupling coefficients of the symmetric group S_k , defined in direct analogy to the case of $SU(2)$. The spins are replaced by irreducible representations of S_k and labelled by Young diagrams with k

boxes. Since, in contrast to $SU(2)$, the Clebsch–Gordan series for S_k is not multiplicity-free, the recoupling coefficients are now linear maps rather than scalars, and we consider their Hilbert–Schmidt norm as a measure for their size, which turns out to be the natural choice in this setup. We then consider the symbols’ norm in the limit where k becomes large but the ratios of the rows in the six Young diagrams stay asymptotically equal. We find that the norm decreases at most polynomially when the ratios converge to the respective ratios of eigenvalues of the reduced density matrices $\rho_A, \rho_B, \rho_C, \rho_{AB}, \rho_{BC}, \rho_{ABC}$ of a tripartite quantum state ρ_{ABC} ; conversely, if there does not exist such a quantum state then the symbols’ norm decreases exponentially (Theorem 1). This description of the asymptotics in terms of the existence of a geometric object (here a quantum state with certain spectral properties) can be seen as a direct analog to the existence of Wigner’s tetrahedra. Our work can also be understood in the context of the quantum marginal problem: we characterize the existence of quantum states of three particles with certain marginal eigenvalues. This extends recent results where it was shown that the case of two particles is guided by the asymptotics of the Kronecker coefficient of S_k [13–17]; indeed, we build on the quantum information methods developed in [13, 18]. Finally, we show that the strong subadditivity of the von Neumann entropy [19] can be deduced solely from symmetry properties of the recoupling coefficients—suggesting that our work might play a role in the current search for new inequalities for the von Neumann entropy.

RECOUPLING COEFFICIENTS

The finite-dimensional irreducible representations of the symmetric group S_k are labelled by *Young diagrams*, that is, ordered partitions $\lambda_1 \geq \dots \geq \lambda_l > 0$ of $\sum_i \lambda_i = k$. We write $\lambda \vdash k$ for such a partition and $[\lambda]$ for the associated irreducible unitary representation of S_k . Every finite-dimensional representation V of S_k can be decomposed into a direct sum of irreducible representations, i.e., there exists an S_k -linear isomorphism $\bigoplus_{\lambda} [\lambda] \otimes H_{\lambda}^V \cong V$, where H_{λ}^V is a vector space whose

dimension is equal to the multiplicity of the irreducible representation $[\lambda]$ in V . If V is a unitary representation then the spaces H_λ^V can be equipped with an inner product such that the isomorphism is unitary. In particular, we may decompose a tensor product $[\alpha] \otimes [\beta]$ of two irreducible representations, resulting in the *Clebsch–Gordan isomorphism*

$$\bigoplus_{\lambda} [\lambda] \otimes H_\lambda^{\alpha\beta} \rightarrow [\alpha] \otimes [\beta]. \quad (1)$$

For a triple tensor product $[\alpha] \otimes [\beta] \otimes [\gamma]$, by associativity there are *two* isomorphic decompositions,

$$\bigoplus_{\lambda, \mu} [\lambda] \otimes H_\lambda^{\mu\gamma} \otimes H_\mu^{\alpha\beta} \cong \bigoplus_{\lambda, \nu} [\lambda] \otimes H_\lambda^{\alpha\nu} \otimes H_\nu^{\beta\gamma}. \quad (2)$$

The *recoupling coefficients* of the symmetric group are the maps

$$\begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}: H_\lambda^{\mu\gamma} \otimes H_\mu^{\alpha\beta} \rightarrow H_{\alpha\nu}^\lambda \otimes H_{\beta\gamma}^\nu \quad (3)$$

defined by embedding $H_\lambda^{\mu\gamma} \otimes H_\mu^{\alpha\beta}$ into $\bigoplus_{\mu} H_\lambda^{\mu\gamma} \otimes H_\mu^{\alpha\beta}$, identifying the latter with $\bigoplus_{\nu} H_\lambda^{\alpha\nu} \otimes H_\nu^{\beta\gamma}$ via (2), and then projecting onto $H_{\alpha\nu}^\lambda \otimes H_{\beta\gamma}^\nu$.

As it is our goal to connect properties of the recoupling coefficients to tripartite quantum states, we will set up a scene in which both the symmetric group and quantum states are at home: the vector space $V = (\mathbb{C}^d)^{\otimes k}$. The symmetric group acts on V by permuting the tensor factors and the special unitary group $SU(d)$ acts diagonally; both actions commute. Therefore, if we decompose V into irreducibles of S_k then the multiplicity spaces—which we shall denote in this case by V_λ^d —are also acted on by the special unitary group. It can be shown that $V_\lambda^d = 0$ when λ has strictly more than d rows, and that it is otherwise an irreducible representation with highest weight λ (the coefficients in a basis of fundamental weights are $\lambda_i - \lambda_{i+1}$). This result is known as Schur–Weyl duality (e.g. [20]) and can be compactly stated in the form:

$$(\mathbb{C}^d)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} [\lambda] \otimes V_\lambda^d \quad (4)$$

In the following we denote by P_λ the orthogonal projectors onto the direct summands.

In order to study the tripartite case, we consider $\mathbb{C}^{abc} \cong \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ and apply Schur–Weyl duality to the k -th tensor powers of \mathbb{C}^a , \mathbb{C}^b and \mathbb{C}^c , separately. Inserting (2), we find

$$\begin{aligned} (\mathbb{C}^{abc})^{\otimes k} &\cong \bigoplus_{\alpha, \beta, \gamma, \mu, \lambda \vdash k} [\lambda] \otimes H_\lambda^{\mu\gamma} \otimes H_\mu^{\alpha\beta} \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c \\ &\cong \bigoplus_{\alpha, \beta, \gamma, \nu, \lambda \vdash k} [\lambda] \otimes H_\lambda^{\alpha\nu} \otimes H_\nu^{\beta\gamma} \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c. \end{aligned} \quad (5)$$

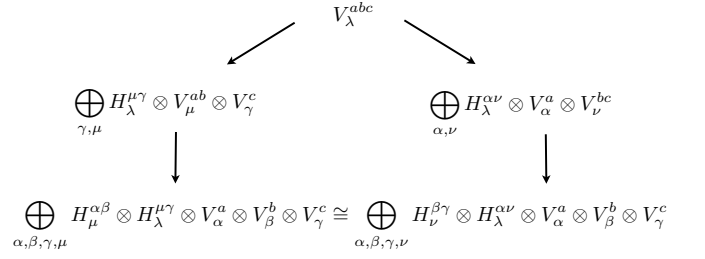


FIG. 1: The recoupling coefficients of the symmetric group can alternatively be defined in terms of the isomorphism connecting the two decompositions of V_λ^{abc} induced by the chains of homomorphisms $SU(a) \times SU(b) \times SU(c) \rightarrow SU(ab) \times SU(c) \rightarrow SU(abc)$ and $SU(a) \times SU(b) \times SU(c) \rightarrow SU(a) \times SU(bc) \rightarrow SU(abc)$.

The associated orthogonal projectors $\tilde{Q} := \tilde{Q}_{\alpha\beta\gamma}^{\lambda, \mu} := (P_\alpha \otimes P_\beta \otimes P_\gamma)(P_\mu \otimes P_\gamma)P_\lambda$ and $\tilde{P} := \tilde{P}_{\alpha\beta\gamma}^{\lambda, \nu} := (P_\alpha \otimes P_\beta \otimes P_\gamma)(P_\alpha \otimes P_\nu)P_\lambda$ satisfy

$$\tilde{P}\tilde{Q} = \mathbf{1}_{[\lambda] \otimes V_\alpha^a \otimes V_\beta^b \otimes V_\gamma^c} \otimes \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}, \quad (6)$$

where both sides are understood as operators on $(\mathbb{C}^{abc})^{\otimes k}$. Therefore,

$$\|\tilde{P}\tilde{Q}\|_\infty \leq \left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}} \leq \text{poly}(k) \|\tilde{P}\tilde{Q}\|_\infty. \quad (7)$$

where for the last inequality we have used the fact that the multiplicity spaces are of dimension at most $\text{poly}(k)$ [31]. Fig. 1 explains an alternative definition of the recoupling coefficients in terms of decomposing the $SU(abc)$ -representation V_λ^{abc} along two chains of group homomorphisms.

THE ASYMPTOTIC LIMIT

For a density operator ρ_{ABC} on $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, we consider the reduced density operators $\rho_{AB} = \text{tr}_C(\rho_{ABC})$, $\rho_A = \text{tr}_{BC}(\rho_{ABC})$ etc., and denote by r_{ABC} , r_{AB} , r_A , etc., the corresponding vectors of eigenvalues (each ordered non-increasingly, e.g. $r_{ABC,1} \geq r_{ABC,2} \geq \dots$). We call the tuple $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ the eigenvalues associated to ρ_{ABC} . The missing r_{AC} will be discussed in the conclusions. Let us define the normalisation of a Young diagram $\lambda \vdash k$ by $\bar{\lambda} := \lambda/k$. The following theorem is the main result of this work:

Theorem 1. *If there exists a quantum state ρ_{ABC} with eigenvalues $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ then there exist Young diagrams $\alpha, \beta, \gamma, \mu, \nu, \lambda \vdash k$ with $k \rightarrow \infty$ such that*

$$\lim_{k \rightarrow \infty} (\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\mu}, \bar{\nu}, \bar{\lambda}) = (r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}) \quad (8)$$

and

$$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}} \geq \frac{1}{\text{poly}(k)}. \quad (9)$$

Conversely, if $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ is not associated to a tripartite density matrix then for every sequence of Young diagrams satisfying (8) we have

$$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}} \leq \exp(-\Omega(k)). \quad (10)$$

Proof. For both directions of the proof we use the spectrum estimation theorem [18] (cf. [13, 21, 22]), which says that k copies of ρ (on \mathbb{C}^d) are mostly supported on the subspaces $[\lambda] \otimes V_\lambda^d$ satisfying $\bar{\lambda} = \lambda/k \approx \text{spec } \rho = r$. More precisely,

$$\text{tr}(P_\lambda \rho^{\otimes k}) \leq \text{poly}(k) \exp(-k \|\bar{\lambda} - r\|_1^2 / 2) \quad (11)$$

where $\|x\|_1 = \sum_i |x_i|$.

We start with the proof of the “if” statement. Define \tilde{Q}_δ as the sum of $\tilde{Q}_{\alpha\beta\gamma}^{\lambda,\mu}$ over those α with $\|\bar{\alpha} - r_A\|_1 \leq \delta$, those β with $\|\bar{\beta} - r_B\|_1 \leq \delta$, etc.; \tilde{P}_δ is defined accordingly. By (11) and the fact that there are only $\text{poly}(k)$ many Young diagrams with a bounded number of rows,

$$\text{tr}(\tilde{P}_\delta \rho_{ABC}^{\otimes k}) \geq 1 - \varepsilon, \quad \text{tr}(\tilde{Q}_\delta \rho_{ABC}^{\otimes k}) \geq 1 - \varepsilon,$$

where $\varepsilon = \text{poly}(k) \exp(-k\delta^2/2)$. Now we use

$$|\text{tr}(PQ\sigma)| \geq \text{tr}(P\sigma) - \sqrt{\text{tr}(\bar{Q}\sigma)} \quad (12)$$

(for pure states $\text{tr}(PQ\sigma) = \langle \phi | PQ | \phi \rangle \geq \langle \phi | P | \phi \rangle - |\langle \phi | \bar{Q} P | \phi \rangle| \geq \langle \phi | P | \phi \rangle - \|\bar{Q} | \phi \rangle\|$; the above follows directly by considering a purification of σ), which holds for arbitrary projectors P and Q and quantum states σ , and obtain

$$|\text{tr}(\tilde{P}_\delta \tilde{Q}_\delta \rho^{\otimes k})| \geq 1 - 2\sqrt{\varepsilon}.$$

Applying the triangle inequality and using (6) and (7), we find that

$$\sum_{\alpha, \beta, \gamma, \mu, \nu, \lambda} \left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}} \geq 1 - 2\sqrt{\varepsilon},$$

where the sum extends over Young diagrams whose normalisation is close to the eigenvalues associated to ρ_{ABC} as specified above. Since the number of terms in the sum can be upper-bounded by $\text{poly}(k)$, we can find sequences of Young diagrams satisfying (8) and (9).

We will now prove the converse statement. Suppose that there exists a sequence of Young diagrams for which (10) does not hold. Passing to a subsequence, we may in fact assume that

$$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}} > \exp(-D_k k) \quad (13)$$

for sequences $k \rightarrow \infty$ and $D_k \rightarrow 0$. Let us choose quantum states $\sigma_{(ABC)^k}$ such that

$$\|\tilde{P}\tilde{Q}\|_\infty^2 = \text{tr}(\tilde{P}\tilde{Q}\sigma_{(ABC)^k}\tilde{Q}\tilde{P}). \quad (14)$$

Both projectors \tilde{P} and \tilde{Q} are S_k -linear, so we may assume $\sigma_{(ABC)^k}$ to be permutation-invariant [32]. Then we can use the bound [23, 24], and find

$$\sigma_{(ABC)^k} \leq \text{poly}(k) \int d\rho_{ABC} \rho_{ABC}^{\otimes k},$$

where $d\rho$ is the Hilbert–Schmidt probability measure on the set of density matrices on \mathbb{C}^{abc} . The right-hand side operator is invariant under unitaries of the form $U^{\otimes k}$, $U \in \text{SU}(abc)$, and therefore commutes with isotypical projectors, so \tilde{Q} can be moved through and, by cyclicity of the trace,

$$\|\tilde{P}\tilde{Q}\|_\infty^2 \leq \text{poly}(k) \int d\rho_{ABC} \text{tr}(\tilde{P}\rho_{ABC}^{\otimes k}\tilde{Q}).$$

It follows, using (7), that there exists a sequence of quantum states $\rho_{ABC,k}$ such that

$$\begin{aligned} \left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}}^2 &\leq \text{poly}(k) \text{tr}(\tilde{P}\rho_{ABC,k}^{\otimes k}\tilde{Q}) \\ &\leq \text{poly}(k) \sqrt{\text{tr}(\tilde{P}\rho_{ABC,k}^{\otimes k})} \sqrt{\text{tr}(\tilde{Q}\rho_{ABC,k}^{\otimes k})}, \end{aligned}$$

where we have used the Cauchy–Schwarz inequality for the second estimate. Recalling the definition of \tilde{P} and \tilde{Q} just above (6), it is easy to see that (13) implies that $\text{tr}(P_\lambda \rho_{ABC,k}^{\otimes k})$, $\text{tr}((P_\mu \otimes \mathbf{1}_c^{\otimes k}) \rho_{ABC,k}^{\otimes k}) = \text{tr}(P_\mu \rho_{AB,k}^{\otimes k})$, etc. all do not decay exponentially with k . Since (13) is stable under passing to a subsequence, the same conclusion holds if we replace the sequence $\rho_{ABC,k}$ by the limit ρ_{ABC} of any of its convergent subsequences (which exist by compactness of the set of density operators). In view of (11), this can only be so if (8) is the tuple of eigenvalues associated with the density matrix ρ_{ABC} . \square

SYMMETRY IMPLIES STRONG SUBADDITIVITY

In the following we use a graphical calculus to deduce a symmetry property of the recoupling coefficients for the symmetric group. Together with the asymptotic limit, this is then shown to imply the strong subadditivity of the von Neumann entropy.

By Schur’s lemma, the multiplicity space $H_\lambda^{\alpha\beta}$ can be identified with the space of S_k -linear maps from $[\lambda]$ to $[\alpha] \otimes [\beta]$. Let us choose a basis ϕ_i satisfying $\text{tr} \phi_j^\dagger \phi_i = \dim[\lambda] \delta_{ij}$, represented by

$$\phi_i = \begin{array}{c} \alpha \quad \beta \\ \swarrow \quad \searrow \\ (i) \\ \uparrow \\ \lambda \end{array} \quad \text{and} \quad \phi_i^\dagger = \begin{array}{c} \alpha \quad \beta \\ \swarrow \quad \searrow \\ (i) \\ \downarrow \\ \lambda \end{array} \quad (15)$$

in the graphical calculus for the fusion basis states (see e.g. [25, 26]). The maps ϕ_i are isometric embeddings onto mutually orthogonal copies of the irreducible representation $[\lambda]$ in $[\alpha] \otimes [\beta]$. By comparing with the definition of the recoupling coefficients, (2) and (3), we find that

$$\mathbf{1}_\lambda \otimes \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}_{ij}^{kl} = \text{Diagram with nodes } l, k, j, i \text{ and arrows } \alpha, \beta, \gamma, \lambda, \mu, \nu.$$

Taking the trace over $[\lambda]$ and deforming the above graphic,

$$\begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}_{ij}^{kl} = \frac{1}{\dim[\lambda]} \text{Diagram with nodes } l, k, j, i \text{ and arrows } \alpha, \beta, \gamma, \lambda, \mu, \nu. \quad (16)$$

The irreducible representations of the symmetric group are self-dual, i.e., $[\lambda] \cong [\lambda]^*$, because they can be defined over the reals. It follows that there exists a single copy of the trivial representation $\mathbf{1}$ in each tensor product $[\lambda] \otimes [\lambda]$, i.e., $H_1^{\lambda\lambda}$ is one-dimensional. We shall denote the corresponding basis vector by

$$\lambda \longleftarrow \bullet \longrightarrow \lambda := \text{Diagram with a circle node and arrows } \lambda, \lambda, \mathbf{1}. \quad (17)$$

It is equal to the maximally entangled state in a real orthonormal basis of $[\lambda]$, i.e., $1/\sqrt{\dim[\lambda]} \sum_{e_\lambda} |e_\lambda\rangle |e_\lambda\rangle$. It is easy to see that we have the teleportation identity

$$\lambda \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \lambda = \frac{1}{\dim[\lambda]} \lambda \longleftarrow \lambda. \quad (18)$$

This leads to the following important property of the Clebsch–Gordan transformation: There exists a unitary matrix U such that

$$\sqrt{\frac{\dim[\alpha] \dim[\beta]}{\dim[\lambda]}} \text{Diagram with nodes } i, i' \text{ and arrows } \alpha, \beta, \lambda = \sum_{i'} U_{ii'} \text{Diagram with nodes } i, i' \text{ and arrows } \lambda, \alpha, \beta. \quad (19)$$

To prove this, it suffices to show that left-hand side maps, which we shall denote by ψ_i , satisfy the same orthogonality relation as the right-hand side basis elements: Indeed, by applying (18) and deforming we find that

$$\text{tr } \psi_j^\dagger \psi_i = \frac{\dim[\alpha] \dim[\beta]}{\dim[\lambda]} \text{Diagram with nodes } i, j \text{ and arrows } \alpha, \beta, \lambda = \frac{\dim[\alpha]}{\dim[\lambda]} \text{Diagram with nodes } i, j \text{ and arrows } \alpha, \beta, \lambda,$$

which is equal to $\dim[\alpha] \delta_{ij}$ in view of the orthogonality properties of the basis elements.

Now we derive the symmetry relation that will play an important role in the proof of strong subadditivity. By (18), the recoupling coefficient (16) is equal to

$$\frac{\dim[\alpha]}{\dim[\lambda]} \text{Diagram with nodes } l, k, j, i \text{ and arrows } \alpha, \beta, \gamma, \lambda, \mu, \nu,$$

from which—by applying (19) and its adjoint to the maps labeled by l and j —we get

$$\frac{\sqrt{\dim[\mu] \dim[\nu]}}{\sqrt{\dim[\beta] \dim[\lambda]}} \sum_{j' l'} \frac{\bar{U}_{jj'} V_{ll'}}{\dim[\nu]} \text{Diagram with nodes } l', k, j', i \text{ and arrows } \alpha, \beta, \gamma, \lambda, \mu, \nu,$$

where U and V are unitaries. Comparing the diagram with the adjoint of (16), we conclude that

$$\left\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \right\|_{\text{HS}} = \sqrt{\frac{\dim[\mu] \dim[\nu]}{\dim[\beta] \dim[\lambda]}} \left\| \begin{bmatrix} \alpha & \mu & \beta \\ \gamma & \nu & \lambda \end{bmatrix} \right\|_{\text{HS}}. \quad (20)$$

This is the crucial symmetry relation satisfied by the recoupling coefficients: by swapping two columns, we pick up a dimension factor according to the corresponding irreducible representations. We remark that this relation has a well-known counterpart for $\text{SU}(2)$: here, the Clebsch–Gordan series is multiplicity-free and (20) holds for the absolute values. The Wigner $6j$ -symbol is then built from the recoupling coefficients precisely in such a way that they are invariant under swapping any two columns, and an analogous definition can be given in our context.

We finally consider a tripartite quantum state ρ_{ABC} . Theorem 1, the polynomial upper bound (7) and the symmetry relation of the recoupling coefficients (20) imply that

$$\frac{\dim[\mu] \dim[\nu]}{\dim[\beta] \dim[\lambda]} \geq \frac{1}{\text{poly}(k)} \quad (21)$$

for a sequence of normalised Young diagrams converging to the respective spectra for the reduced density matrices. Since for large k , $\frac{1}{k} \log_2 \dim[\lambda] \rightarrow H(r) = \sum_i -\bar{\lambda}_i \log_2 \bar{\lambda}_i$, we conclude that the von Neumann entropy is strongly subadditive [19],

$$H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_B) + H(\rho_{ABC}) \quad (22)$$

for all quantum states ρ_{ABC} . Weak monotonicity, $H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_A) + H(\rho_C)$, follows similarly by swapping the columns $(\alpha, \gamma) \leftrightarrow (\mu, \nu)$ instead of $(\beta, \lambda) \leftrightarrow (\mu, \nu)$ in (20).

CONCLUSIONS

In the spirit of Wigner’s seminal work on the semiclassical limit of the $6j$ -symbols, we have given a geometric characterization to the asymptotic behavior of the recoupling coefficients of the symmetric group: it is determined by the existence of tripartite quantum states with certain marginal spectra. Our methods directly generalise to higher recoupling coefficients (the analogs of general Wigner $3nj$ -symbols) and quantum states of several particles: just as Theorem 1 characterizes six of the seven marginal spectra (with r_{AC} missing), in general a linear number of marginal spectra can be fixed—suggesting that higher-order representation-theoretic structures might play a role in the characterization of all marginal spectra. In this sense, our result may be regarded as a partial quantum-mechanical version of Chan and Yeung’s description of the set of local Shannon entropies in terms of sizes of Young subgroups [27]. We hope that our work might provide some useful perspective in the search for new entropy inequalities for the von Neumann entropy, the “laws of quantum information theory” [28–30].

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- [31] Here and throughout this text, $\text{poly}(k)$ denotes polynomials in k that depend only on a , b and c —the dimensions of the local Hilbert spaces \mathbb{C}^a , \mathbb{C}^b and \mathbb{C}^c , respectively.
- [32] A k -partite density matrix ρ is called permutation-invariant if $\pi \rho \pi^{-1} = \rho$ for all $\pi \in S_k$.

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